

# Hopf-Galois module structure of tame radical extensions of prime degree

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Hopf algebras and Galois module structure

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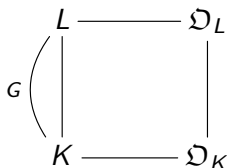
# Overview

This talk is about the application of Hopf algebras to questions of integral Galois module structure in finite extensions of number fields.

- Integral Galois module structure
- Tame Kummer extensions of prime degree
- Hopf-Galois structures on field extensions
- Tame radical extensions of prime degree

## Integral Galois module structure

Let  $L/K$  be a finite Galois extension of number fields with group  $G$ . Write  $\mathfrak{D}_L, \mathfrak{D}_K$  for the rings of algebraic integers of  $L, K$ .



$L$  is a free  $K[G]$ -module of rank one.

Study  $\mathfrak{D}_L$  as a module over its *associated order* in  $K[G]$

$$\mathfrak{A}_{K[G]} = \{z \in K[G] \mid z \cdot \mathfrak{D}_L \subseteq \mathfrak{D}_L\}.$$

We have  $\mathfrak{D}_K[G] \subseteq \mathfrak{A}_{K[G]}$ , with equality if and only if  $\text{Tr}_{L/K}(\mathfrak{D}_L) = \mathfrak{D}_K$ .  
(That is: if and only if  $L/K$  is *tame*.)

## Integral Galois module structure

### Theorem (Noether, 1932)

*If  $L/K$  is tame then  $\mathfrak{D}_L$  is a locally free  $\mathfrak{D}_K[G]$ -module (of rank one).*

That is:  $\mathfrak{D}_{K,\mathfrak{p}} \otimes \mathfrak{D}_L$  is a free  $\mathfrak{D}_{K,\mathfrak{p}}[G]$ -module for each prime ideal  $\mathfrak{p}$  of  $\mathfrak{D}_K$ .

### Theorem (Hilbert-Speiser, 1897, 1916)

*If  $L/\mathbb{Q}$  is tame and abelian then  $\mathfrak{D}_L$  is a free  $\mathbb{Z}[G]$ -module.*

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### Theorem (Hilbert-Speiser, 1897, 1916)

*If  $L/\mathbb{Q}$  is tame and abelian then  $\mathfrak{D}_L$  is a free  $\mathbb{Z}[G]$ -module.*

For base fields different from  $\mathbb{Q}$  the problem is more delicate.

### Theorem (Greither et al, 1999)

*If  $K \neq \mathbb{Q}$  then there exists a prime number  $p$  and a tame Galois extension  $L/K$  of degree  $p$  such that  $\mathfrak{D}_L$  is not a free  $\mathfrak{D}_K[G]$ -module.*

# Tame Kummer extensions of degree $p$

## Theorem (Gómez Ayala, 1994)

Let  $K$  be a number field and  $p$  be a prime number. Suppose that  $\zeta_p \in K$  and that  $L/K$  is a tame Galois extension of degree  $p$  with group  $G$ . Then  $\mathfrak{D}_L$  is a free  $\mathfrak{D}_K[G]$ -module if and only if there exists an element  $\beta \in \mathfrak{D}_L$  such that

- 1  $L = K(\beta)$ ,
- 2  $b = \beta^p \in \mathfrak{D}_K$ ,
- 3 the ideals  $\mathfrak{b}_j$  defined by  $\mathfrak{b}_j = \prod_{\mathfrak{p}} \mathfrak{p}^{\lfloor v_{\mathfrak{p}}(b^j)/p \rfloor}$  for  $j = 0, 1, \dots, p-1$  are

principal, with generators  $b_j$  such that  $\sum_{j=0}^{p-1} \frac{\beta^j}{b_j} \equiv 0 \pmod{p\mathfrak{D}_L}$ .

# A new proof of Gómez Ayala's result

## Theorem (Bley and Johnston, 2007)

Let

- $L/K$  be a Galois extension of number fields with group  $G$ ;
- $\mathfrak{M}$  be a maximal order in  $K[G]$  that contains  $\mathfrak{A}_{K[G]}$ .

Then  $\mathfrak{D}_L$  is a free  $\mathfrak{A}_{K[G]}$ -module if and only if

- $\mathfrak{D}_L$  is a locally free  $\mathfrak{A}_{K[G]}$ -module;
- $\mathfrak{M}\mathfrak{D}_L$  is a free  $\mathfrak{M}$ -module, with a generator  $x \in \mathfrak{D}_L$ .

## A new proof of Gómez Ayala's result

Now suppose that  $L/K$  is a tame Kummer extension of prime degree  $p$ .

- $\mathfrak{D}_L$  is a locally free  $\mathfrak{D}_K[G]$ -module by Noether's theorem.
- We have  $K[G] \cong K^p$  via orthogonal idempotents.
- $K[G]$  contains a unique maximal order  $\mathfrak{M} \cong \mathfrak{D}_K^p$ .
- $\mathfrak{M}\mathfrak{D}_L$  is a free  $\mathfrak{M}$ -module if and only if there exists an integral Kummer generator  $\beta$  whose associated ideals  $\mathfrak{b}_j$  are principal.
- $\mathfrak{M}\mathfrak{D}_L$  has a generator in  $\mathfrak{D}_L$  if and only if there exists  $\beta$  as above whose associated ideals have generators satisfying the congruence in point (3) of the theorem.



## Hopf-Galois structures

Let  $L/K$  be a finite Galois extension of fields with group  $G$ .

The action of  $K[G]$  on  $L$  is an example of a *Hopf-Galois structure* on the extension; there may be others.

If  $H$  is a Hopf algebra giving a Hopf-Galois structure on  $L/K$  then define

$$\mathfrak{A}_H = \{h \in H \mid h \cdot \mathfrak{D}_L \subseteq \mathfrak{D}_L\},$$

and study the structure of  $\mathfrak{D}_L$  as a module over the various  $\mathfrak{A}_H$ .

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**Proposition (Childs, 1989/Kohl, 1998/Byott, 1996)**

If  $L/K$  is a Galois extension of prime degree  $p$  then the Hopf-Galois structure given by  $K[G]$  is the only Hopf-Galois structure on  $L/K$ .

# Hopf-Galois structures

Non-normal extensions of number fields may also admit Hopf-Galois structures.

These provide frameworks in which we can study the rings of integers of such extensions.

## Idea

Let  $K$  be a number field, suppose that  $\zeta_p \notin K$ , and let  $L/K$  be a radical extension of degree  $p$  (that is:  $L = K(\alpha)$  with  $\alpha^p \in K - K^p$ ).

Can we use Hopf-Galois structures to study  $\mathfrak{O}_L$ ?

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Can we use Hopf-Galois structures to study  $\mathfrak{O}_L$ ?

Proposition (Childs, 1989/Kohl, 1998/Byott, 1996)

The extension  $L/K$  admits a unique Hopf-Galois structure.

# Main Result

## Theorem

Let  $K$  be a number field and  $p$  be a prime number. Suppose that  $p$  is unramified in  $K$  and that  $L/K$  is a tame radical extension of degree  $p$ . Let  $H$  give the unique Hopf-Galois structure on  $L/K$ . Then

- $\mathfrak{D}_L$  is a locally free  $\mathfrak{A}_H$ -module;
- $\mathfrak{D}_L$  is a free  $\mathfrak{A}_H$ -module if and only if there exists  $\beta \in \mathfrak{D}_L$  such that

①  $L = K(\beta),$

②  $b = \beta^p \in \mathfrak{D}_K,$

③ the ideals  $\mathfrak{b}_j$  defined by  $\mathfrak{b}_j = \prod_p \mathfrak{p}^{\lfloor v_p(b^j)/p \rfloor}$  for  $j = 0, 1, \dots, p-1$  are

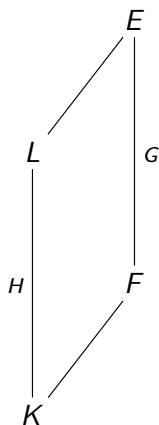
principal, with generators  $b_j$  such that  $\sum_{j=0}^{p-1} \frac{\beta^j}{b_j} \equiv 0 \pmod{p\mathfrak{D}_L}.$

## Sketch of the proof

- We have  $H \cong K^P$  via orthogonal idempotents.
- $H$  contains a unique maximal order  $\mathfrak{M} \cong \mathfrak{D}_K^P$ .
- Bley and Johnston:  $\mathfrak{D}_L$  is a free  $\mathfrak{A}_H$ -module if and only if it is a locally free  $\mathfrak{A}_H$ -module and  $\mathfrak{M}\mathfrak{D}_L$  is free on an element of  $\mathfrak{D}_L$ .
- $\mathfrak{D}_L$  is a locally free  $\mathfrak{A}_H$ -module:
  - If  $\mathfrak{p} \nmid \mathfrak{p}\mathfrak{D}_K$  then  $\mathfrak{A}_{H,\mathfrak{p}} = \mathfrak{M}_{\mathfrak{p}}$ , so  $\mathfrak{D}_{L,\mathfrak{p}}$  is a free  $\mathfrak{A}_{H,\mathfrak{p}}$ -module.
  - If  $\mathfrak{p} \mid \mathfrak{p}\mathfrak{D}_K$  then compute explicit  $\mathfrak{D}_{K,\mathfrak{p}}$  bases of  $\mathfrak{D}_{L,\mathfrak{p}}$  and  $\mathfrak{A}_{H,\mathfrak{p}}$ , and show that  $\mathfrak{D}_{L,\mathfrak{p}}$  is a free  $\mathfrak{A}_{H,\mathfrak{p}}$ -module.
- $\mathfrak{M}\mathfrak{D}_L$  is a free  $\mathfrak{M}$ -module if and only if the ideals  $\mathfrak{b}_j$  are principal.
- $\mathfrak{M}\mathfrak{D}_L$  has a generator in  $\mathfrak{D}_L$  if and only if the congruence in point (3) of the theorem is satisfied.

## Is this the best result?

- Continue to suppose that
  - $K$  is a number field;
  - $p$  is unramified in  $K$ ;
  - $L/K$  is a tame radical extension of degree  $p$ ;
  - $H$  gives the unique Hopf-Galois structure on  $L/K$ .
- Let  $F = K(\zeta_p)$  and  $E = FL$ .
- Then  $E/F$  is a Kummer extension of degree  $p$ ; let  $G = \text{Gal}(E/F)$ .
- Our results imply that if  $\mathfrak{D}_L$  is a free  $\mathfrak{A}_H$ -module then  $\mathfrak{D}_E$  is a free  $\mathfrak{D}_F[G]$ -module.
- Is the converse true?



## Where next?

- Tame extensions of degree  $p^2$ :
  - Radical:  $L = K(\alpha)$  with  $\alpha^{p^2} \in K - K^{p^2}$ .  
Harder to characterize tame versions of these.
  - Biradical:  $L = K(\alpha, \beta)$  with  $\alpha^p, \beta^p \in K - K^p$ .  
Hopf-Galois structures not known in this case.
- Wild extensions of degree  $p$ :
  - There is a unique Hopf-Galois structure.
  - If  $\mathfrak{p} \nmid p\mathfrak{D}_K$  then  $\mathfrak{D}_{L,\mathfrak{p}}$  is a free  $\mathfrak{A}_{H,\mathfrak{p}}$ -module.
  - What about for  $\mathfrak{p} \mid p\mathfrak{D}_K$ ?  
Elder has analysed the Hopf-Galois module structure of wild non-normal extensions of local fields of degree  $p$ . Perhaps his results could be used to complete the local picture.
  - Then use result of Bley and Johnston to get global results.



Thank you for your attention.